

Revisiting metric perturbations in tensor-vector-scalar theory

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The cosmological behavior of modified gravity theories with additional degrees of freedom (DOFs) is typically complex and can give rise to non-intuitive results. A possible way of exploring such theories is to consider appropriate parameterizations of these new DOFs. Here I suggest using the algebraic structure of trivial identities, which typically occur at the level of the perturbed field equations, for defining such parameterizations. Choosing the example of Bekenstein's tensor-vector-scalar theory (TeVeS) and considering perturbations in the conformal Newtonian gauge, this parameterization is then used to study several aspects of the cosmological evolution in an Einstein-de Sitter universe. As a main result, I conclude that perturbations of the scalar field take a key role in generating enhanced growth if this enhancement is primarily associated with a gravitational slip. From this point of view, the previously found modified growth in TeVeS is truly a result of the complex interplay between both the scalar and the vector field. Since the occurrence of trivial identities of the above kind appears as a generic feature of modified gravity theories with extra DOFs, these parameterizations should generally prove useful to investigate the cosmological properties of other proposed modifications. As such parameterizations capture the full nature of modifications by construction, they also provide a suitable framework for developing semi-analytic models of cosmologically interesting quantities like, for instance, the growth factor, leading to various applications. Supplementary to numerical analysis, parameterizations based on trivial identities are thus an interesting tool to approach modified gravity theories with extra DOFs.

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I. INTRODUCTION

As is well known, the current concordance model of cosmology has proven quite successful in reproducing observations on the largest physical scales [1]. Based on general relativity (GR), this model heavily relies on two phenomenologically motivated ingredients, namely cold dark matter (CDM) and dark energy (DE), which are believed to account for approximately 95% of the Universe's energy-matter content. However, there are still several problems. From a more fundamental point of view, for instance, the introduction of DE remains theoretically challenging and appears extremely fine-tuned, despite the various proposals for its dynamics [2]. The CDM paradigm, on the other hand, struggles with several issues as well. Apart from the lack of direct experimental detection [3], these include the question of its cosmological abundance [4] and problems related to the formation of structure on small scales [5] which are still subject to intensive debates in the literature.

Given this rather unsatisfactory situation, one might take a completely different view on the above. One possibility is to consider that the gravitational description genuinely differs from GR, mimicking the basic effects of the dark components and alleviating some, if not all, of the problems inherent to the standard framework. Additionally motivated by considerations beyond the field of cosmology, approaches into this direction have recently (re-)gained some interest in the scientific community and there exists now a plethora of modified gravity theories which could serve as an alternative to GR. Among many other proposals [6], examples of such modifications include conformal Weyl gravity [7], $f(R)$ gravity [8]

or Brans-Dicke theory [9]. In general, modifications of the gravitational sector come at the cost of simplicity, typically giving rise to complex and less transparent theories that exhibit a non-intuitive behavior and are difficult to work with. This is especially true if a modified framework introduces new independent degrees of freedom (DOFs), for instance, by considering (non-minimal) couplings to additional fields such as scalars, vectors or higher-order tensors. Thus it is often hard to get a grasp on the detailed features and underlying mechanisms of such theories which are usually investigated with the help of numerical methods. Considering the limitations of a numerical treatment, i.e. the restriction to a finite set of theory parameters and initial conditions, it is only natural to look for alternative and supplementary ways of analyzing the impact of modifications on the cosmological evolution. A possible approach into this direction is to work out suitable parameterizations of the new DOFs arising in a certain theory or generalized classes thereof. In particular, this might not only help to obtain a better understanding of a theory's intrinsic structure, but also to highlight potential problems and limitations, which could provide a toehold for design improvements. It is almost needless to say that, for this purpose, such parameterizations are required to be as close as possible to the actually assumed model. Note that this ansatz differs in motivation from the recent attempts of parameterizing deviations from GR in a model-independent fashion [10] which are interesting in their own right.

Here I want to develop the above idea in the context of Bekenstein's tensor-vector-scalar theory (TeVeS) [11] and use it to reinvestigate its behavior on cosmological scales. Considering perturbations around a spatially flat Friedmann-Robertson-Walker (FRW) background, I will present an approach toward parameterizing the new perturbation variables which is based on the occurrence of trivial identities at the level of the field equations. For the conformal Newtonian

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gauge, this parameterization is then used to study several aspects of the cosmological evolution such as the modified growth of density perturbations and the gravitational slip which have been identified as rather generic features of modified gravity theories [12]. For simplicity and to allow an analytic treatment of the background quantities, I will restrict the present analysis to an Einstein-de Sitter (EdS) universe which contains pressureless matter only. While this will suffice to gain a qualitative understanding of the perturbation evolution, it also provides an excellent approximation to the matter era of a realistic universe.

In its first proposed form, TeVeS has been constructed as a relativistic extension for the modified Newtonian dynamics (MOND) paradigm [13] which aims at solving the missing mass problem on galactic scales by postulating an acceleration-dependent change of Newton's law. As such, TeVeS has been designed as a theory that is significantly different from GR. Analyzing the rotation curves of spiral galaxies, Milgrom estimated the characteristic scale of this transition as $a_0 \approx 1.2 \times 10^{-10} \text{ m s}^{-2}$, and it is interesting to note that MOND still appears to be in good agreement with much of the available extragalactic data [14]. As it turns out, the phenomenology associated with this kind of modifications and its generalizations [15] is quite rich. It is therefore not too surprising that TeVeS and related theories have been subject to many different studies in the literature [16] which range from the Solar System up to cosmological volumes. Note that, despite several inconsistencies arising from its construction, e.g. in the strong gravity regime [17], the original formulation of TeVeS provides a viable theoretical framework for cosmological models [18].

The paper is organized as follows: Starting with a brief review of TeVeS and its relation to MOND in Sec. II, the corresponding cosmological background together with the modified EdS model are discussed in Sec. III. Considering perturbations in the conformal Newtonian gauge, I then introduce the parameterization of the new perturbation variables and investigate the cosmological perturbation evolution in Sec. IV. Finally, I conclude in Sec. V. For clarity, some of the material involving lengthy expressions is presented in an appendix. To aid the reader through the technical parts, two tables of symbols are presented. Table I summarizes all relevant quantities and parameters except for the ones related to perturbations which are separately listed in Table II. Throughout this work, I will assume a positive metric signature (+2) and use natural units, i.e. units where the speed of light, the reduced Planck constant and the Boltzmann constant are all equal to unity. If not stated otherwise, greek and latin indices run from 0 to 3 and from 1 to 3, respectively.

II. FUNDAMENTALS OF TEVES

A. Fields and action

TeVeS [11, 19] has originally been proposed as a bimetric theory of gravity which is based on three dynamical fields: an

Einstein metric $\tilde{g}_{\mu\nu}$, a time-like vector field A_μ such that

$$\tilde{g}^{\mu\nu} A_\mu A_\nu = -1, \quad (1)$$

and a scalar field ϕ . Furthermore, there is a second metric $g_{\mu\nu}$ which is needed for gravity-matter coupling only and obtained from the non-conformal relation

$$g_{\mu\nu} = e^{-2\phi} \tilde{g}_{\mu\nu} - 2A_\mu A_\nu \sinh(2\phi). \quad (2)$$

The frames delineated by the metric fields $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ will be called *Einstein frame* and *matter frame*, respectively. The geometric part of the action is exactly the same as in GR:

$$S_g = \frac{1}{16\pi G} \int \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} \sqrt{-\tilde{g}} d^4x, \quad (3)$$

where $\tilde{R}_{\mu\nu}$ is the Ricci tensor of $\tilde{g}_{\mu\nu}$ and \tilde{g} the determinant of $\tilde{g}_{\mu\nu}$. Note that the TeVeS constant G must not be mistaken for the Newtonian gravitational constant G_N (see Sec. II B). The vector field's action S_v reads as follows:

$$S_v = -\frac{1}{32\pi G} \int [K_B F^{\mu\nu} F_{\mu\nu} - \lambda(A_\mu A^\mu + 1)] \sqrt{-\tilde{g}} d^4x, \quad (4)$$

with $F_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu$ and indices being raised and lowered with respect to $\tilde{g}_{\mu\nu}$, i.e. $A^\mu = \tilde{g}^{\mu\nu} A_\nu$. Here the constant K_B describes the coupling of the vector field to gravity and λ is a Lagrangian multiplier enforcing the normalization condition given by Eq. (1). Equation (4) corresponds to the classical Maxwell action, the field A_μ now having an effective mass. The action S_s of the scalar field ϕ involves an additional non-dynamical scalar field μ , and takes the form

$$S_s = -\frac{1}{16\pi G} \int [\mu h^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi + V(\mu)] \sqrt{-\tilde{g}} d^4x, \quad (5)$$

where $h^{\mu\nu} = \tilde{g}^{\mu\nu} - A^\mu A^\nu$ and $V(\mu)$ is an initially arbitrary (potential) function. As the field μ is related to the invariant $h^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi$, however, it could in principle be eliminated from the action. Finally, matter is required to obey the weak equivalence principle, and thus the matter action is given by

$$S_m = \int \mathcal{L}_m[g, \Upsilon^B, \nabla \Upsilon^B] \sqrt{-g} d^4x, \quad (6)$$

where Υ^B is a generic collection of matter fields. Note that world lines are by construction geodesics of the metric $g_{\mu\nu}$ rather than $\tilde{g}_{\mu\nu}$. As usual, the corresponding equations of motion can be derived by varying the total action $S = S_g + S_v + S_s + S_m$ with respect to the basic fields. Despite its explicit bimetric construction, TeVeS may be written in pure tensor-vector form [20] and provides a particular example of generalized (extended) Aether-type theories [21].

B. Choice of the potential, quasistatic systems and relation to MOND

Whether or not TeVeS recovers the dynamics of MOND in the nonrelativistic limit depends on the assumed form of the

potential V . Originally, Bekenstein made the choice

$$V(\mu) = \frac{3\mu_0^2}{128\pi l_B^2} \left[\hat{\mu} (4 + 2\hat{\mu} - 4\hat{\mu}^2 + \hat{\mu}^3) + 2 \log(1 - \hat{\mu})^2 \right], \quad (7)$$

where the constant l_B corresponds to a length scale, $\hat{\mu} = \mu/\mu_0$ and μ_0 is a dimensionless constant [22]. One can show that the framework of GR is then recovered in the limit $K_B \rightarrow 0$ and $l_B \rightarrow \infty$. To obtain the theory's nonrelativistic limit, one may apply the usual approximations for weak fields and quasistatic systems. In this case, one has $V' \equiv dV/d\mu < 0$, and therefore $0 < \mu < \mu_0$. Using that also $V(\mu) < 0$ for the given range, the resulting metric $g_{\mu\nu}$ turns out to be basically identical to the metric obtained in GR if the nonrelativistic gravitational potential is replaced by

$$W = \Xi \Phi_N + \phi, \quad (8)$$

$$\Xi = e^{-2\phi_C} (1 + K_B/2)^{-1},$$

where ϕ_C is the cosmological value of ϕ at the time the system in question breaks away from the cosmological expansion, and Φ_N is the Newtonian potential generated by the matter density ρ [23]. In this approximation, it is consistent to assume that A_μ is pointing into the direction of the timelike Killing vector associated with the static spacetime. Then we have

$$h^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi \rightarrow (\nabla \phi)^2 \equiv \|\nabla \phi\|_2^2 \quad (9)$$

and the equation of the scalar field reduces to

$$\nabla \cdot (\mu \nabla \phi) = 8\pi G \rho. \quad (10)$$

As has been shown in Ref. [11], Eqs. (8) and (10) correspond to the MOND paradigm: If $\mu \rightarrow \mu_0$, the theory reaches its (exact) Newtonian limit, and the measured gravitational constant G_N is given by

$$G_N = \frac{\mu_0 + 2 - K_B}{\mu_0(1 - K_B/2)} G. \quad (11)$$

Similarly, the theory reaches its MONDian limit as $\mu \rightarrow 0$ and the acceleration constant a_0 can be expressed in terms of the TeVeS and potential parameters,

$$a_0 = \frac{\sqrt{6}}{2l_B} \frac{e^{\phi_C}}{\sqrt{\pi\mu_0}} \frac{G}{G_N}. \quad (12)$$

As can be seen from above, a_0 depends on ϕ_C and may therefore, in principal, change with time. For viable cosmological models (see Sec. III), however, such changes are expected to be basically imperceptible [24]. Note that requiring the emergence of MONDian dynamics fixes the potential only asymptotically and leaves substantial freedom for constructing particular models.

It should be pointed out that potentials like the one specified in Eq. (7) exhibit a disconnection between the regimes relevant for cosmology and quasistatic systems, respectively. Since cosmological models require $V' \geq 0$ [11], one obtains $\mu > \mu_0$ and thus cannot use the same potential branch as for

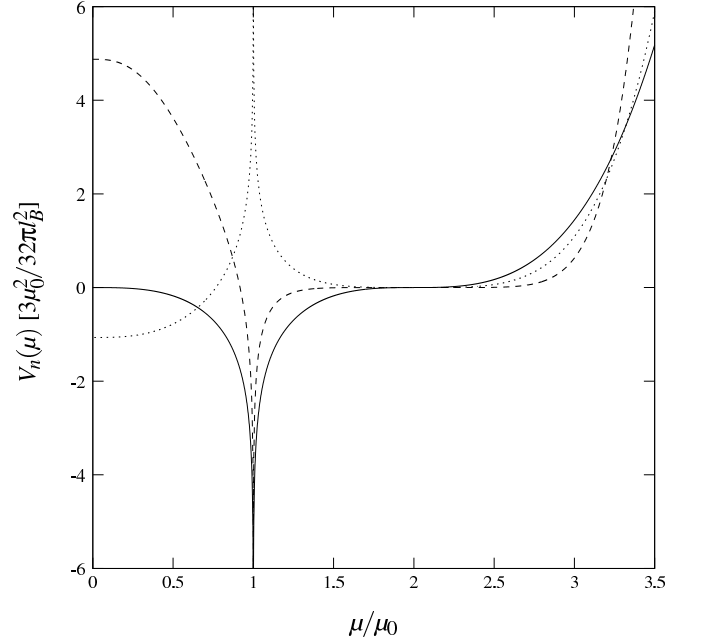


FIG. 1. Illustration of the generalized potential function $V_n(\mu)$ given by Eq. (13) for $n = 2$ (solid line), 3 (dotted line), and 6 (dashed line).

quasistatic systems ($\mu < \mu_0$). Lacking a smooth transition between these two regimes, however, it is unclear how bound systems such as galaxies would decouple from the Hubble flow or if such a decoupling results in the quasistatic limit discussed above. To resolve this issue, an interesting alternative has been proposed in [25], with its cosmology studied in Ref. [26]. In the present work, however, I will only focus on the theory's cosmological behavior and thus not follow this approach.

Instead - for reasons that will become clear below - I will assume the following general class of potentials throughout this paper [27]:

$$V_n(\mu) = \frac{3\mu_0^2}{32\pi l_B^2} \left[\frac{n+4+(n+1)\hat{\mu}}{(n+1)(n+2)} (\hat{\mu}-2)^{n+1} + \frac{(-1)^n}{2} \log(1-\hat{\mu})^2 + \sum_{m=1}^n \frac{(-1)^{n-m}}{m} (\hat{\mu}-2)^m \right], \quad (13)$$

where $n \geq 2$ [28]. Adopting different values of n , Fig. 1 illustrates the resulting potential shape as a function of $\hat{\mu}$. Note that the such generalized potential reduces to Bekenstein's toy model if $n = 2$. The derivative of $V_n(\mu)$ takes a simpler form and can be expressed as

$$V'_n(\mu) = \frac{3\mu_0}{32\pi l_B^2} \hat{\mu}^2 \frac{(\hat{\mu}-2)^n}{\hat{\mu}-1}. \quad (14)$$

Remembering that cosmological models must satisfy the condition $V' \geq 0$ and requiring that V' is single-valued, one is always free to choose between two possible potential branches. In accordance with previous investigations [11, 27, 29], I shall use the branch ranging from the extremum at $\mu = 2\mu_0$ to in-

finitly. Under these preliminaries, it was found that the potential in Eq. (13) gives rise to tracker solutions of the scalar field [27], with a background evolution similar to other general cosmological theories involving tracker fields [30]. I will further elaborate on this behavior and an approximate analytic treatment in Sec. III B.

III. COSMOLOGICAL BACKGROUND

A. Evolution equations

Imposing the usual assumptions of an isotropic and homogeneous spacetime, both $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are given by FRW metrics with scale factors a and $b = a \exp \bar{\phi}$, respectively, where $\bar{\phi}$ is the background value of the scalar field [11, 31]. Adopting a spatially flat universe, the modified Friedmann equation in the matter frame reads

$$3H^2 = 8\pi G_{\text{eff}} (\bar{\rho}_\phi + \bar{\rho}), \quad (15)$$

where the physical Hubble parameter is $H = \dot{a}/a^2$ and the overdot denotes the derivative with respect to conformal time. Here $\bar{\rho}$ corresponds to the FRW background density of the fluid and the scalar field density takes the form

$$\bar{\rho}_\phi = \frac{e^{2\bar{\phi}}}{16\pi G} (\bar{\mu} V' + V). \quad (16)$$

The effective gravitational coupling strength is given by

$$G_{\text{eff}} = G e^{-4\bar{\phi}} \left(1 + \frac{d\bar{\phi}}{d \log a} \right)^{-2} \quad (17)$$

which is generally time-varying through its dependence on the scalar field $\bar{\phi}$. Just as in GR, the energy density $\bar{\rho}$ evolves according to

$$\dot{\bar{\rho}} = -3 \frac{\dot{a}}{a} (1 + w) \bar{\rho}, \quad (18)$$

where w is the equation-of-state (EoS) parameter of the fluid. In case of multiple background fluids, i.e. $\bar{\rho} = \sum_i \bar{\rho}_i$, the relative densities Ω_i are defined as

$$\Omega_i = 8\pi G_{\text{eff}} \frac{\bar{\rho}_i}{3H^2} = \frac{\bar{\rho}_i}{\bar{\rho} + \bar{\rho}_\phi}. \quad (19)$$

The evolution of the scalar field ϕ is governed by

$$\ddot{\bar{\phi}} = \dot{\bar{\phi}} \left(\frac{\dot{a}}{a} - \dot{\bar{\phi}} \right) - \frac{1}{U} \left[3\bar{\mu} \frac{b}{b} \dot{\bar{\phi}} + 4\pi G a^2 e^{-4\bar{\phi}} (\bar{\rho} + 3\bar{P}) \right], \quad (20)$$

where \bar{P} is the fluid's background pressure, and the function U is related to the potential V ,

$$U(\bar{\mu}) = \bar{\mu} + 2 \frac{V'}{V''}. \quad (21)$$

In addition, the scalar field obeys the constraint equation

$$\dot{\bar{\phi}}^2 = \frac{1}{2} a^2 e^{-2\bar{\phi}} V' \quad (22)$$

which can be inverted to obtain $\bar{\mu}(a, \bar{\phi}, \dot{\bar{\phi}})$. For later use, I also introduce the relation

$$2 \frac{\dot{a}}{a} \frac{\dot{b}}{b} - \frac{\ddot{b}}{b} - \bar{\mu} \dot{\bar{\phi}}^2 = 4\pi G a^2 e^{-4\bar{\phi}} (\bar{\rho} + \bar{P}) \quad (23)$$

which follows from combining Eq. (15) with Eq. (22) and the corresponding Raychaudhuri equation [31].

According to previous investigations [11, 27, 29], a broad range of expressions for the potential V , including the choice in Eq. (13), leads to $\exp \bar{\phi} \approx 1$ and $\bar{\rho}_\phi \ll 1$ throughout cosmological history. Therefore, the background evolution will be very similar to the standard case of GR, with only small corrections induced by the scalar field.

B. Tracker solutions of the scalar field

For the class of potentials specified in Eq. (13), it has been found that the scalar field exhibits a (stable) tracking behavior and synchronizes its energy density with the dominant component of the universe [27, 29]. Tracking occurs as V' tends to its zero point where $\bar{\mu} = 2\mu_0$, and the evolution of the field $\bar{\phi}$ during tracking is approximately given by

$$\bar{\phi} = \bar{\phi}_0 + \frac{|1 + 3w|}{2\beta\mu_0|1 - w| - |1 + 3w|} \log a, \quad (24)$$

where $\bar{\phi}_0$ is an integration constant and $\beta = \pm 1$, with the actual sign depending on the background fluid's EoS parameter w and Eq. (20). Its density $\bar{\rho}_\phi$ then exactly scales like that of the fluid, and the relative density parameter Ω_ϕ turns approximately into a constant,

$$\Omega_\phi = \frac{(1 + 3w)^2}{6\mu_0(1 - w)^2}. \quad (25)$$

Note that the right-hand side of Eq. (24) slightly differs from the expression presented in Ref. [27]. In Appendix A, I discuss why this is the case and show that Eq. (24) is indeed the correct result.

Following the lines of Ref. [27], $\bar{\mu}$ may then be expressed as $\bar{\mu} = 2\mu_0(1 + \epsilon)$ with $0 < \epsilon \ll 1$. Using $V'(2\mu_0) = 0$ and expanding V' to lowest order in ϵ , Eq. (22) leads to

$$\epsilon = \frac{1}{2} \left(\frac{16\pi l_B^2}{3\mu_0} \frac{e^{2\bar{\phi}}}{a^2} \dot{\bar{\phi}}^2 \right)^{1/n}. \quad (26)$$

It turns out that this is the only stage at which the constant l_B enters the evolution equations. In preparation for Sec. IV, I further take the time derivative of the above, which yields the useful relation

$$\dot{\bar{\phi}} \dot{\epsilon} = \frac{2}{n} \left(\frac{\dot{\bar{\phi}}^2}{\bar{\phi}} - \dot{\bar{\phi}} \frac{\dot{a}}{a} + \ddot{\bar{\phi}} \right) \epsilon. \quad (27)$$

Note that stable tracking requires ϵ to asymptotically decrease to zero, i.e. $\epsilon \rightarrow 0$. Therefore one has the condition $\dot{\epsilon} < 0$ which may be used to infer the proper sign of the parameter β in Eq. (24) (see Appendix A).

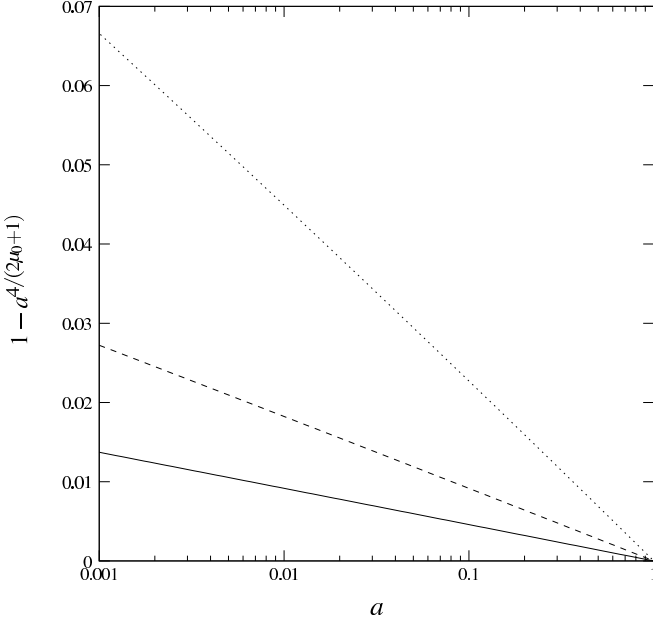


FIG. 2. Relative deviation of the Hubble expansion in the modified EdS cosmology from the ordinary GR case: Shown are the results for $\mu_0 = 200$ (dotted line), 500 (dashed line), and 1000 (solid line).

C. Modified Einstein-de Sitter cosmology

In what follows, I shall assume a universe entirely made of pressureless matter with perfect tracking of the scalar field, corresponding to the EdS model in GR. Setting $\bar{P} = w = 0$ fixes $\beta = -1$, and thus the scalar field can be written as

$$\bar{\phi} = \bar{\phi}_0 - \frac{1}{2\mu_0 + 1} \log a. \quad (28)$$

To find the proper value of β , one may either insert Eq. (24) into Eq. (20), or use the argument presented in Appendix A. Since the fluid evolves according to Eq. (18), the density takes the form $\bar{\rho} = \bar{\rho}_0 a^{-3}$, where $\bar{\rho}_0$ is the background density's value today. Thus exploiting Eq. (25) allows one to rewrite the modified Friedmann equation in the matter frame as

$$H^2 = H_0^2 a^{-3+4/(2\mu_0+1)}, \quad (29)$$

where I have used the definition

$$H_0^2 = e^{-4\bar{\phi}_0} \frac{8\pi G \bar{\rho}_0}{3} \left(1 + \frac{1}{6\mu_0 - 1}\right) \left(1 - \frac{1}{2\mu_0 + 1}\right)^{-2}. \quad (30)$$

From Eq. (29), it is evident that the deviation of the Hubble expansion from the ordinary EdS case is entirely characterized by the parameter μ_0 . For several reasons [11], μ_0 should take a rather large value on the order of 100 – 1000, and thus this deviation will be very small. Assuming different choices of μ_0 , Fig. 2 shows the relative difference between the models as a function of the scale factor a , indicating that the change of the expansion is only at the percent level.

IV. METRIC PERTURBATIONS IN TEVES

A. Preliminaries

1. Matter-frame perturbations

Now I will turn to metric perturbations around a spatially flat FRW spacetime in TeVeS. The starting point is the set of linear perturbation equations for TeVeS which have been derived in fully covariant form in Ref. [31]. For simplicity, I shall restrict the analysis to scalar modes only and work within the conformal Newtonian gauge. In this case, metric perturbations are characterized by two scalar potentials Ψ and Φ , and the line element in the matter frame is given by

$$ds^2 = a^2 \left[-(1 + 2\Psi) d\tau^2 + (1 - 2\Phi) \delta_{ij} dx^i dx^j \right]. \quad (31)$$

Similarly, one needs to consider perturbations of the other fields: While the fluid perturbation variables are defined in the usual way, i.e. the density perturbation, for instance, is expressed in terms of the density contrast δ ,

$$\rho = \bar{\rho} + \delta\rho = \bar{\rho}(1 + \delta), \quad (32)$$

the scalar field is perturbed as

$$\phi = \bar{\phi} + \varphi, \quad (33)$$

where φ is the scalar field perturbation. Finally, the perturbed vector field is written as

$$A_\mu = ae^{-\bar{\phi}} (\bar{A}_\mu + \alpha_\mu), \quad (34)$$

where $\bar{A}_\mu = (1, 0, 0, 0)$ and

$$\alpha_\mu = (\Psi - \varphi, \nabla\alpha). \quad (35)$$

Note that the time component of the vector field perturbation is constrained to be a combination of metric and scalar field perturbations, which is a consequence of the unit-norm condition in Eq. (1). Therefore, one needs to consider only the longitudinal perturbation component α .

2. Einstein-frame perturbations

Instead of using Eq. (31), one may also express perturbations in the Einstein frame [29, 31]. In this case, metric perturbations are written as

$$\tilde{g}_{00} = -b^2 e^{-4\bar{\phi}} (1 + 2\tilde{\Psi}), \quad (36)$$

$$\tilde{g}_{0i} = -b^2 \partial_i \tilde{\zeta}, \quad (37)$$

$$\tilde{g}_{ij} = b^2 (1 - 2\tilde{\Phi}) \delta_{ij}. \quad (38)$$

In terms of matter-frame variables, the Einstein-frame perturbations are given by

$$\tilde{\Psi} = \Psi - \varphi, \quad (39)$$

$$\tilde{\Phi} = \Phi - \varphi, \quad (40)$$

$$\tilde{\zeta} = (e^{-4\bar{\phi}} - 1)\alpha. \quad (41)$$

To avoid lengthy expressions in the perturbed field equations, it is convenient to work with variables from both frames. As I will frequently use these equations in the following sections, the full set of linear perturbation equations in the conformal Newtonian gauge is, for clarity, given in Appendix B.

Since the equation governing the evolution of the scalar field perturbation is of second order, it is further helpful to introduce an auxiliary field γ which allows one to split the scalar field equation into a system of two first-order equations [31]. Performing this split for the present gauge choice, the field γ is given by the relation (see Appendix B)

$$\gamma = \mathcal{F}(\bar{q}_i, \dot{\varphi}, \ddot{\Psi}), \quad (42)$$

where the function \mathcal{F} has the form specified in Eq. (B4) and the \bar{q}_i denote a collection of background quantities and their derivatives with respect to conformal time.

B. Parameterizing the new degrees of freedom

To begin with, one may take a naive approach and try to relate the new perturbation variables to perturbations of the metric or the matter fluid by a suitable combination of the field equations. Considering the perturbation equations in Appendix B and assuming a general matter fluid whose background evolution is given by Eq. (18), I take the time derivative of Eq. (B8) and eliminate $\dot{\theta}$ with the help of Eq. (B2) from the resulting expression. The next step is to get rid of the time derivatives of $\ddot{\Phi}$. This can be achieved by exploiting an algebraic relation which is obtained from combining Eqs. (B9) and (B10). Finally, using Eqs. (39) and (B4), one arrives at the equation

$$\begin{aligned} & \left[2\frac{\dot{a}}{a}\frac{\dot{b}}{b} - \frac{\ddot{b}}{b} - \ddot{\mu}\dot{\phi}^2 - 4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+w) \right] \ddot{\Psi} \\ &= \left[\ddot{\mu} \left(4\dot{\phi}^2 + 2\dot{\phi}\frac{\dot{a}}{a} + \ddot{\phi} \right) + \ddot{\mu}\dot{\phi} + 4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+3w) \right] \dot{\varphi}. \end{aligned} \quad (43)$$

From Eq. (27), one immediately sees that the coefficient in front of $\ddot{\Psi}$ vanishes. Thus the above gives a trivial identity and one cannot infer any information on the relation between the scalar field perturbation φ and the metric potential $\ddot{\Psi}$. This somewhat reflects the fact that φ corresponds to a full DOF in the theory and the occurrence of trivial relations like in Eq. (43) is indeed a generic feature of modified gravity theories which introduce additional DOFs [32].

The idea is now to use the algebraic structure of these trivial identities as a starting point for defining a suitable parameterization. For purposes that will become clear below, consider a new function B_φ such that

$$\ddot{\Psi} = B_\varphi \varphi. \quad (44)$$

In general, B_φ will change with time and exhibit scale-dependence which, for instance, may come as a ratio to some preferred scale(s) of the theory. Moreover, it will also depend on the used cosmological background, the theory parameters, the scalar potential $V(\mu)$, and the choice of initial conditions.

As for the perturbation variables γ and $\tilde{\zeta}$, it is possible to motivate a relation similar to the above. To see this, I multiply Eq. (B7) with the scale factor b and take the time derivative. Combining the result with Eqs. (B5) and (B10), I eliminate \dot{E} and the time derivative of $\tilde{\zeta}$, respectively. Substituting $\dot{\varphi}$ and $\dot{\gamma}$ with help of Eqs. (B3) and (B4), respectively, one eventually ends up with

$$\begin{aligned} & ae^{-\bar{\phi}} \left[3\frac{\dot{b}}{b}\frac{\dot{\mu}}{U}\dot{\phi} - \frac{\dot{a}}{a}\dot{\phi} + \dot{\phi}^2 + \ddot{\phi} + \frac{1}{U}4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+3w) \right] \gamma \\ & - 6\frac{\dot{b}}{b} \left[\ddot{\mu}\dot{\phi} + \ddot{\mu}\dot{\phi} + 2\left(\dot{\phi} + \frac{\dot{b}}{b}\right)\ddot{\mu}\dot{\phi} + 4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+3w) \right] \varphi \\ & - 2 \left[2\frac{\dot{b}}{b}\left(\dot{\phi} - \frac{\dot{b}}{b}\right) + \frac{\ddot{b}}{b} + \ddot{\mu}\dot{\phi}^2 + 4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+w) \right] \\ & \times \left(3\dot{\Phi} + k^2 \tilde{\zeta} + 3\frac{\dot{b}}{b} \ddot{\Psi} \right) = 0, \end{aligned} \quad (45)$$

where \mathbf{k} is the conformal wave vector and $k = |\mathbf{k}|$. Using the background relations presented in Sec. III A, one finds that Eq. (45) again yields a trivial identity, with a form very similar to that found before. A direct comparison between Eqs. (43) and (45) suggests the definition of another function B_γ which relates γ to $\tilde{\zeta}$ and the time derivative of $\ddot{\Phi}$. A suitable expression may be obtained by formally subtracting Eq. (44) from Eq. (45) and “normalizing” the resulting equation, leading to the ansatz

$$3\dot{\Phi} + k^2 \tilde{\zeta} + \frac{ae^{-\bar{\phi}}}{2U} B_\gamma \gamma = 0. \quad (46)$$

The expressions given by Eqs. (44) and (46) provide formal closure relations which may be used to eliminate perturbations of the scalar and vector field from the evolution equations (see Appendix C for details). Thus perturbations to the new DOFs are genuinely absorbed into the functions B_φ and B_γ . While the function B_φ simply describes the magnitude of scalar field perturbations relative to the metric potential $\ddot{\Psi}$, the interpretation of B_γ is less obvious. As the term $k^2 \tilde{\zeta}$ appears in its definition, however, one may expect that B_γ characterizes the impact of vector field perturbations. As will become clear in the following sections, this is to some extent the case and the current parameterization disentangles the effects of the vector and scalar fields. Although the functions B_φ and B_γ are entirely determined by choosing a specific model and a set of initial conditions, one may view them as “free parameters” describing modifications induced by the additional DOFs. Concerning the analysis presented below, I will assume that B_φ and B_γ are regular and require that $B_\varphi, B_\gamma \neq 0$ at all cosmological times and scales, which is necessary to exclude pathological cases.

Although I have only considered the situation within the conformal Newtonian gauge, note that the occurrence of trivial identities is gauge-independent, and thus there is always the possibility of constructing similar parameterizations for different gauge choices. As parameterizations of this kind could prove useful for other modified theories, it might further be interesting to adopt a gauge-invariant approach toward such parameterizations, which I leave for future work.

C. Subhorizon scales

In the following, I shall assume the previously discussed modified EdS cosmology with perfect tracking of the scalar field. This allows one to use the corresponding background expressions presented in Sec. III C and considerably simplifies the analysis of the modified equations. Since $\bar{\mu}$ resides close to its minimum in this case, i.e. $\bar{\mu} = 2\mu_0(1 + \epsilon)$ with $\epsilon \ll 1$, one may further exploit the two first-order expressions

$$\frac{\bar{\mu}}{U} = 1 - \frac{2}{n}\epsilon \quad (47)$$

and

$$\frac{2\mu_0}{U} = 1 - \frac{n+2}{n}\epsilon, \quad (48)$$

which is useful to rewrite terms involving the field U .

1. Modified potentials

Adopting an EdS universe together with the closure relations presented in Sec. IV B, one may now write metric perturbations solely in terms of the matter fluid variables, B_φ and B_γ . The resulting expressions are quite lengthy and can be found in Appendix C. As a first application, I shall investigate the theory's behavior on scales much smaller than the horizon where $aH/k \ll 1$. To allow further progress, I will additionally assume that the typical time variation of B_φ and B_γ is comparable or smaller than that of the Hubble expansion, i.e.

$$\left| \frac{\dot{B}_\varphi}{B_\varphi} \right| \lesssim aH, \quad \left| \frac{\dot{B}_\gamma}{B_\gamma} \right| \lesssim aH, \quad (49)$$

such that the contribution of $|\dot{B}_\varphi/B_\varphi|$ and $|\dot{B}_\gamma/B_\gamma|$ compared to k is much smaller than unity [33]. Inserting the logarithmic approximation for $\bar{\phi}$ specified by Eq. (28) and using the assumptions given in Eq. (49), I expand the corresponding equations for the Einstein-frame potentials $\tilde{\Psi}$ and $\tilde{\Phi}$ in powers of aH/k . To lowest order, this yields

$$\tilde{\Psi} = -\tilde{A} \frac{a^2 H^2}{k^2} \delta, \quad (50)$$

$$\tilde{\Phi} = -\tilde{B} \frac{a^2 H^2}{k^2} \delta, \quad (51)$$

where

$$\tilde{A} = \frac{e^{4\bar{\phi}} B_\varphi [2(B_\gamma + 2\mu_0)(e^{4\bar{\phi}} - 1) + K_B B_\gamma]}{2(B_\gamma + 2B_\varphi)(e^{4\bar{\phi}} - 1) + K_B [B_\gamma - 2B_\varphi(e^{4\bar{\phi}} - 1)]} \times \frac{6\mu_0 - 1}{(2\mu_0 + 1)^2} + O(\epsilon) \quad (52)$$

and

$$\tilde{B} = \frac{e^{4\bar{\phi}} [2\mu_0(B_\gamma + 2B_\varphi)(e^{4\bar{\phi}} - 1) + K_B B_\gamma B_\varphi e^{4\bar{\phi}}]}{2(B_\gamma + 2B_\varphi)(e^{4\bar{\phi}} - 1) + K_B [B_\gamma - 2B_\varphi(e^{4\bar{\phi}} - 1)]} \times \frac{6\mu_0 - 1}{(2\mu_0 + 1)^2} + O(\epsilon). \quad (53)$$

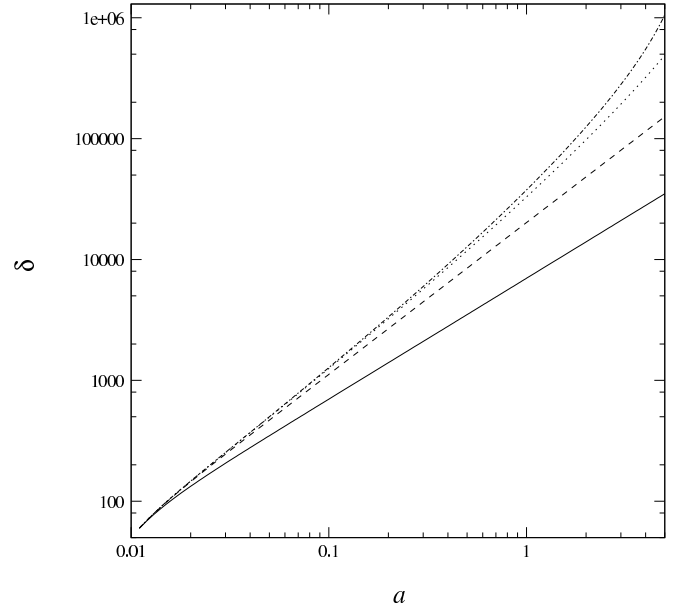


FIG. 3. Evolution of subhorizon density perturbations in the modified EdS cosmology: Assuming the model parameters specified in the text with $B_\varphi = B_\gamma = 3\mu_0/2$, the figure illustrates results for $K_B = 1$ (dashed line), 0.1 (dotted line), and 0.09 (dashed-dotted line). For comparison, the corresponding evolution in the ordinary EdS model is also shown (solid line).

Note that although I have omitted the resulting expressions to first order in ϵ for clarity, their contribution, which typically amounts to a correction of a few percent or less, will fully be taken into account when conducting calculations.

Compared to the situation in GR, where $\tilde{A} = \tilde{B} = 3/2$ for the EdS model, the above immediately shows that the potentials exhibit a more sophisticated time dependence. This dependence does not only involve the perturbation quantities B_φ and B_γ , but also the background scalar field $\bar{\phi}$ which is therefore expected to have a significant impact on the potentials' evolution and the growth of density perturbations.

2. Growth of density perturbations

Equipped with an analytic expression for the the potential $\tilde{\Psi}$ (or equivalently Ψ), I now proceed with the analysis of structure growth in the context of TeVeS. As is well known, the ordinary EdS model in GR gives rise to a growth equation of the form

$$\frac{d^2 \delta}{da^2} + \frac{3}{2a} \frac{d\delta}{da} - \frac{3}{2a^2} \delta = 0, \quad (54)$$

with the two solutions $\delta \propto a^{-3/2}$ and $\delta \propto a$. Following the same derivation as in GR, the TeVeS analog of Eq. (54) for the present assumptions reads

$$\frac{d^2 \delta}{da^2} + \frac{1}{2a} \left(3 + \frac{4}{2\mu_0 + 1} \right) \frac{d\delta}{da} - \frac{\tilde{A}}{a^2} (1 + B_\varphi^{-1}) \delta = 0, \quad (55)$$

where \tilde{A} is given by Eq. (52) and $\bar{\phi}$ evolves according to Eq. (28). It is instructive to further simplify the expression for \tilde{A} by exploiting that the background field $\bar{\phi}$ is much smaller than unity, i.e. $|\bar{\phi}| \ll 1$. A straightforward expansion immediately yields

$$\tilde{A} \approx \frac{3B_\varphi}{2\mu_0} + 6\frac{B_\varphi}{\mu_0} \left[1 + 2\frac{B_\gamma}{B_\gamma} - \frac{4(B_\varphi - \mu_0)}{K_B B_\gamma} \right] \bar{\phi} + O(\bar{\phi}^2), \quad (56)$$

where I have additionally neglected terms proportional to ϵ and used that $\mu_0 \gg 1$. As is obvious from the above, the vector field enters the growth equation not only through B_γ , but also implicitly through K_B .

Adopting the potential specified in Eq. (7), i.e. the $n = 2$ case, numerical studies of the TeVeS cosmology revealed that structure is allowed to form more efficiently than in the framework of GR [29]. A later analysis of this behavior has identified vector perturbations and their associated instability as the key ingredient for the found enhanced growth, where it has further been shown that this enhancement only occurs if the coupling constant K_B is chosen small enough [34, 35]. On the other hand, it has been argued that scalar field perturbations play only a negligible role for structure formation, with φ oscillating around a value roughly μ_0 times smaller than metric perturbations during the matter era. Here I seek a detailed understanding of this enhanced growth mechanism based on the modified growth equation and the functions B_φ and B_γ .

In accordance with the above findings, I will assume from here on that B_φ takes values on the order of μ_0 , i.e. $B_\varphi \sim \mu_0$. To start with a simple case, consider first the situation that \tilde{A} is dominated by the zeroth-order term such that $\tilde{A} \approx 3B_\varphi/2\mu_0$, and that $B_\varphi \approx \text{const}$. Then Eq. (55) can be solved analytically and the growing mode approximately evolves as a power law, $\delta \propto a^p$ with

$$p = \frac{1}{4} \left(\sqrt{1 + 24\frac{B_\varphi}{\mu_0}} - 1 \right), \quad (57)$$

where the ordinary EdS behavior is obviously recovered if $B_\varphi = \mu_0$. Next, one may ask how the first-order contributions to \tilde{A} modify the growth of density perturbations. To answer this question, one first observes that, since $B_\varphi \sim \mu_0$ and $|\bar{\phi}| \ll 1$, the relevant terms in Eq. (56) are those depending on the function B_γ . Clearly, these terms can have a significant impact on \tilde{A} only if either B_γ or the product $K_B B_\gamma$ acquire sufficiently small values which are able to cancel the suppression by $\bar{\phi}$. The resulting modification, i.e. an effective increase or decrease in growth, then depends on the actual numbers that are (or need to be) plugged in.

To further simplify the present analysis, assume that any relevant first-order contribution in Eq. (56) comes solely from the term proportional to K_B^{-1} . This property may formally be implemented by imposing the additional condition $|B_\gamma| \gtrsim B_\varphi$. As will become clear in the following section, such an assumption is not unreasonable if the modification of growth is associated with a gravitational slip, i.e. a significant difference between the two metric potentials Ψ and Φ . For the same reason (see Sec. IV C 3), I will here exclude the case $B_\varphi = \mu_0$

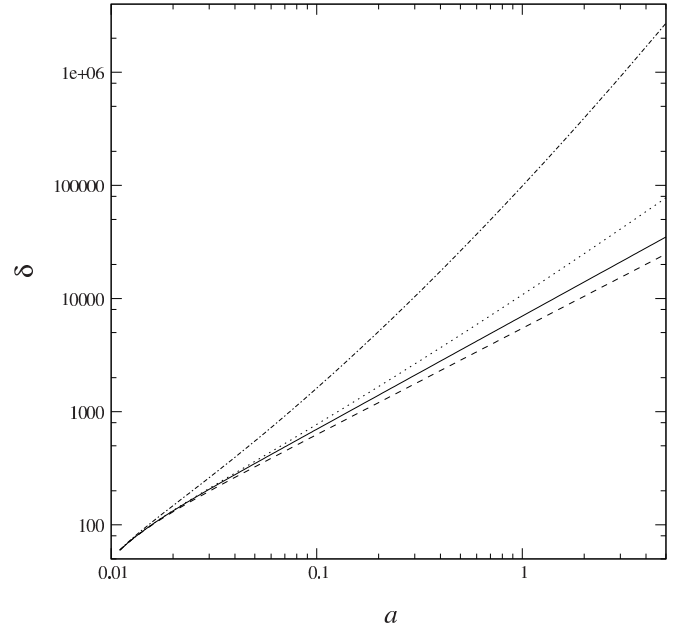


FIG. 4. Same as Fig. 3, but now assuming a configuration with $B_\varphi = 9\mu_0/10$ and $B_\gamma = -9\mu_0/5$: Shown are results for $K_B = 1$ (dashed line), 0.005 (dotted line), and 0.001 (dashed-dotted line), together with the ordinary EdS case (solid line).

for which the first-order term under consideration vanishes. Then, for a fixed sign of $\bar{\phi}$ and $B_\varphi \neq \mu_0$, the form of Eq. (56) suggests that there are two principle configurations which can lead to an increase of growth. Choosing $\bar{\phi} < 0$, for instance, one finds that either $B_\varphi > \mu_0$ and $B_\gamma > 0$ or $B_\varphi < \mu_0$ and $B_\gamma < 0$ are required to obtain the desired enhancement (Note that a small negative value for the field $\bar{\phi}$ is in harmony with the results of Ref. [34] and, contrary to previous claims, does not automatically violate causality [36]).

To demonstrate that this behavior truly emerges from the full potential expression in Eq. (52), one may numerically solve Eq. (55) for different sets of trial parameters. Since the aim here is to develop a qualitative understanding, I will consider the case that both B_φ and B_γ are approximately constant [37]. Assuming a $B_\gamma > 0$ scenario with $B_\varphi = B_\gamma = 3\mu_0/2$ and setting $\mu_0 = 1000$, $l_B = 100\text{Mpc}$, $\bar{\phi}_0 = -0.003$, the Hubble constant $H_0 = 100\text{ km/s/Mpc}$, and $n = 4$ for the scalar potential, Fig. 3 illustrates the numerically calculated evolution of δ for different values of K_B and an arbitrary, but fixed choice of initial conditions at $a = 0.01$. As can be seen from the figure, this simplistic model recovers an increased growth for small values of K_B ($\lesssim 0.1$). For large values of K_B ($\gtrsim 1$), however, this enhancement does not occur and the density contrast follows a power law, with an index p given by Eq. (57). Similar results can be obtained for $B_\gamma < 0$, and leaving the other parameters unchanged, an example for $B_\varphi = 9\mu_0/10$ and $B_\gamma = -2B_\varphi = -9\mu_0/5$ is shown in Fig. 4.

While both configurations, i.e. $B_\gamma > 0$ or $B_\gamma < 0$, can give rise to increased growth, they substantially differ in the way this growth is sourced by the potential $\tilde{\Psi}$. If $B_\gamma > 0$, models of the above kind suggest that the potential $\tilde{\Psi}$ evolves

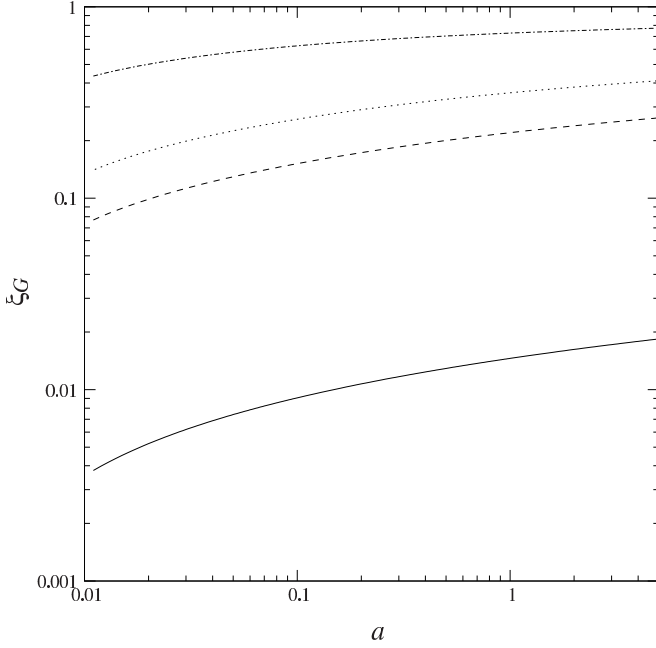


FIG. 5. Gravitational slip associated with the simplistic growth model shown in Fig. 4: The figure illustrates results for $K_B = 1$ (solid line), 0.01 (dashed line), 0.005 (dotted line), and 0.001 (dashed-dotted line).

in an exponential fashion. This, in turn, leads to a faster and faster increase of the density contrast as seen in Fig. 3. What is disturbing is that all such models with $\bar{\phi} < 0$ eventually run into a singularity, which appears as a consequence of the denominator in Eq. (52). Although one may imagine that realistic time-varying choices of B_ϕ and B_γ avoid a singularity, this result provides a strong indication against the $B_\gamma > 0$ case. On the other hand, models with $B_\gamma < 0$ are generally free of this problem and associated with a slower, logarithmic growth of the potential. Using the gravitational slip in the following section, it will turn out that the enhanced growth reported in Ref. [34] does indeed correspond to the situation where $B_\gamma < 0$.

3. Gravitational slip

To conclude the discussion of subhorizon scales, I will briefly demonstrate how the previously discussed mechanism giving rise to enhanced growth generates differences between the matter-frame potentials Ψ and Φ . Remember that in GR, such a difference can only be caused by anisotropic stress of the matter fluid which is basically negligible for the cosmological evolution at late times. For this purpose, it is useful to introduce a quantity ξ_G characterizing the difference between the two metric potentials. Here I choose the definition

$$\xi_G \equiv \frac{\Psi - \Phi}{\Psi} = \frac{B_\phi(\tilde{\Psi} - \tilde{\Phi})}{(1 + B_\phi)\tilde{\Psi}}. \quad (58)$$

Assuming that $B_\phi \sim \mu_0 \gg 1$ and $|\bar{\phi}| \ll 1$ (see Sec. IV C 2), the above expression becomes

$$\xi_G \approx -4 \left[1 - \frac{2(B_\phi - \mu_0)}{K_B B_\phi} \right] \bar{\phi} + O(\bar{\phi}^2), \quad (59)$$

where I have again neglected terms of order ϵ . Interestingly, the first-order contribution in Eq. (59) does not depend on the function B_γ and contains a term proportional to $K_B^{-1}(B_\phi - \mu_0)$, strongly resembling that previously found in the expansion of the function \tilde{A} . Now consider the case $B_\phi = \mu_0$ for which this contribution disappears. From Eqs. (52) and (53), it then follows that the zeroth-order expressions for \tilde{A} and \tilde{B} are almost identical, only differing by a factor of $\exp(4\bar{\phi})$ which appears in the second term of the numerator. Consequently, the relative difference between the metric potentials will only involve terms on the order of $\bar{\phi}$ and ϵ , becoming vanishingly small independent of what is assumed for B_γ . Therefore, if the enhanced growth is primarily sourced or driven by the gravitational slip, this heavily suggests that one must have $B_\phi \neq \mu_0$ and picks out the term $\propto K_B^{-1}$ in Eq. (56) as the predominant contribution, in accordance with the growth analysis presented in the previous section.

Since the sign of ξ_G eventually depends on whether B_ϕ takes values that are smaller or larger than μ_0 , one may further use the gravitational slip to infer the sign of B_γ and thus the proper growth configuration from the numerical results obtained in Ref. [34]. There it has been found that ξ_G significantly grows during the matter era with $\xi_G > 0$ and $\bar{\phi} < 0$ [38]. If one uses Eq. (59), this leads to $B_\phi < \mu_0$ and thus $B_\gamma < 0$ as increased growth would not occur otherwise (again, see Sec. IV C 2). As such, the additional growth previously found in Ref. [29] generally corresponds to the situation of the simplistic model depicted in Fig. 4. Assuming different values of K_B , the slip parameter ξ_G associated with this particular model is illustrated in Fig. 5.

From the findings of Sec. IV C 2 and the above discussion, it becomes clear that the modified growth in TeVeS is truly a result of the complex interplay between the newly introduced DOFs, i.e. both the scalar and the vector field [39]. While the scalar field enters the growth equation through B_ϕ and its background value $\bar{\phi}$, the vector field affects the evolution of δ through perturbations, i.e. the function B_γ , and the coupling constant K_B . If the observed enhanced growth is mostly sourced by the gravitational slip, perturbations of the scalar field take a key role in the growth mechanism and need to evolve in a way such that B_ϕ takes values sufficiently different from μ_0 .

D. Superhorizon scales

Similar to the previous section, it is possible to analyze the theory on scales much larger than the horizon. In this case, terms proportional to k^2 may safely be neglected in the perturbation equations, i.e. $k \rightarrow 0$. Using Eqs. (B7) and (B1), one

thus starts from

$$-3\frac{\dot{b}}{b}\left(\dot{\tilde{\Phi}} + \frac{\dot{b}}{b}\tilde{\Psi}\right) + \frac{ae^{-\tilde{\Phi}}}{2}\dot{\tilde{\Phi}}\gamma = 4\pi Ga^2 e^{-4\tilde{\Phi}}\bar{\rho}(\delta - 2\varphi) \quad (60)$$

and

$$\dot{\delta} = 3\dot{\Phi} = 3\left(\dot{\tilde{\Phi}} + \dot{\varphi}\right). \quad (61)$$

As usual, the above implies that the combination $\delta - 3\Phi$ is conserved over time. Using Eqs. (60) and (61) together with the remaining perturbation equations and the closure relations from Sec. IV B, it is possible to arrive at an equation governing the evolution of the Einstein-frame potential $\tilde{\Phi}$. The corresponding derivation is sketched in Appendix C 2, and the final expression takes the form

$$\ddot{\tilde{\Phi}} + C_\Phi \frac{\dot{a}}{a} \dot{\tilde{\Phi}} = 0, \quad (62)$$

where the coefficient C_Φ depends on background quantities, the parameter functions B_φ and B_γ , and their time derivatives (again, see Appendix C 2). Since there appear only derivatives of the metric potential $\tilde{\Phi}$ in Eq. (62), one immediately obtains the solution $\tilde{\Phi} = \text{const}$. The second solution is determined by the evolution of C_Φ in the modified EdS cosmology, but finding it requires detailed knowledge about the functional form of both B_φ and B_γ . To get some more insight, however, consider the situation that the time derivatives of these functions are approximately negligible. In this case, one can show that the coefficient C_Φ approximately turns into

$$C_\Phi \approx \frac{B_\varphi - 2B_\gamma + 1}{2B_\varphi - B_\gamma + 2}, \quad (63)$$

where I have again assumed that $\mu_0 \gg 1$ and $\epsilon \ll 1$. The corresponding solution is then given by a power law, $\tilde{\Phi} \propto a^p$, with index

$$p = -\frac{3}{2} \frac{B_\gamma}{B_\gamma - 2(B_\varphi + 1)}. \quad (64)$$

The nature of this solution is obviously fixed by the actual configuration of the functions B_φ and B_γ . To end up with a decaying mode, for instance, Eq. (64) yields the condition $2(B_\varphi + 1)/B_\gamma < 1$. Remembering that scalar field perturbations should appear suppressed compared to the metric potentials, i.e. $|B_\varphi| \gg 1$, this is typically satisfied if B_φ and B_γ are of opposite sign, but there is also substantial freedom for other configurations. In particular, Eq. (64) reduces to the ordinary GR case if $B_\gamma = 5(B_\varphi + 1)$. Taking the view that TeVeS modifies GR without introducing any very radical changes in the principal behavior of this solution, it seems reasonable to assume that it will generally correspond to a decaying mode for viable and realistic choices of C_Φ . If this holds true, one is left with the result that the Einstein-frame potential $\tilde{\Phi}$ is frozen.

As can be seen from Eq. (61), however, this result does not necessarily apply to the matter-frame potential Φ and the evolution of the density contrast which additionally involve the scalar field perturbation φ . Following a similar procedure

TABLE I. Summary of relevant generic symbols and parameters: Note that barred symbols denote the corresponding background quantities in the FRW spacetime, and an additional subscript “0” indicates the value of a background quantity today ($a = 1$).

Symbol	Description
$g_{\mu\nu}$	Metric tensor in the matter frame
$\tilde{g}_{\mu\nu}$	Metric tensor in the Einstein frame
a	Scale factor in the matter frame
b	Scale factor in the Einstein frame, $b = ae^{\tilde{\Phi}}$
ϕ	TeVes scalar field
A^μ	TeVes vector field
$F_{\mu\nu}$	Field strength tensor of the TeVeS vector field A^μ
μ	Auxiliary nondynamical scalar field
$V(\mu)$	Free potential function of the scalar field μ
$U(\mu)$	Function of μ related to V , $U = \mu + 2V'/V''$
ρ	Density of the matter fluid
P	Pressure of the matter fluid
w	Matter fluid's equation-of-state parameter, $w = P/\rho$
H	Physical Hubble parameter
G	Bare gravitational constant
G_{eff}	Effective gravitational coupling for FRW dynamics
K_B	Coupling constant of the TeVeS vector field A^μ
$V_n(\mu)$	Generalized Bekenstein potential, see Eq. (13)
n	Integer characterizing the potential V_n , $n \geq 2$
μ_0	Dimensionless parameter of the potential V_n
l_B	Overall length scale entering the potential V_n
ϵ	Offset of $\bar{\mu}/2\mu_0$ from minimum during tracking, $\epsilon \ll 1$

as for Eq. (62), it is also possible to obtain an equation for φ which takes the form of a simple mechanical oscillator, i.e.

$$\ddot{\varphi} + d_1\dot{\varphi} + d_2\varphi = 0. \quad (65)$$

Inserting the background relations of the modified EdS cosmology and using $\mu_0 \gg 1$, the coefficients in Eq. (65) can be expressed as

$$d_1 \approx \left(2 + \frac{B_\varphi}{2\mu_0}\right)\frac{\dot{a}}{a}, \quad d_2 \approx \frac{\dot{B}_\varphi}{2\mu_0}\frac{\dot{a}}{a} \quad (66)$$

to lowest order in ϵ . In general, one has $d_1, d_2 \neq 0$, and thus solutions to Eq. (65) will vary over time. Therefore, making use of Eq. (61), the potential Φ and δ are not strictly conserved on superhorizon scales. The actual behavior of the perturbation φ is determined by the quantity $d_1^2 - 4d_2$, leading to damped oscillations or exponentially growing and decaying solutions. Given the typically found magnitudes of scalar field perturbations [29, 34, 35], however, one may expect a rather small impact on the evolution of Φ and δ , but this is generally subject to initial conditions. Finally, note that such solutions with $\dot{\varphi} \neq 0$ also indicate an evolution of the Einstein-frame potential $\tilde{\Psi}$. This can be seen from

$$\ddot{\varphi} + 2\frac{\dot{a}}{a}\dot{\varphi} + \frac{1}{2\mu_0}\frac{\dot{a}}{a}\dot{\tilde{\Psi}} \approx 0, \quad (67)$$

which follows from combining Eqs. (44), (65) and (66).

V. CONCLUSIONS

The cosmological behavior of modified gravity theories with extra DOFs is typically complex and can give rise to non-intuitive results. A possible approach toward exploring such theories and their modifications is to consider suitable parameterizations of these new DOFs. In this work, I have suggested using the algebraic structure of trivial identities, which typically occur at the level of the perturbed field equations, as a starting point for defining such parameterizations. Choosing the specific example of Bekenstein's TeVeS theory and considering only scalar modes in the conformal Newtonian gauge, I have used this ansatz to construct formal closure relations for the additional perturbation variables arising from the extra DOFs. These relations introduce two new auxiliary perturbation quantities, the parameter functions B_φ and B_γ , which allow one to re-express perturbations of the TeVeS scalar and vector fields in the evolution equations and prove useful when investigating different cosmological aspects of the theory.

Assuming the modified EdS background cosmology, i.e. the background solution of a universe dominated by pressureless matter together with the cosmological scalar field acting as a perfect tracker field, I have studied in detail the theory's behavior on subhorizon scales. Focusing on the evolution of density perturbations, the present parameterization has been adopted to derive the TeVeS analog of the well-known growth equation and to address the question of how the additional DOFs affect the dynamics of the density contrast δ . Identifying the relevant terms of the TeVeS growth mechanism, I have explicitly demonstrated which configurations can lead to an increased growth rate and discussed their connection to the gravitational slip, i.e. a significant difference between the two metric potentials. As a main result, I conclude that perturbations of the scalar field take a key role in generating enhanced growth if this enhancement is primarily associated with a gravitational slip. From this point of view, the previously found modified growth in TeVeS is truly a result of the complex interplay between both the scalar and the vector field. Considering the limit of superhorizon scales, it can be argued that non-decaying solutions of the Einstein-frame potential $\tilde{\Phi}$ are exactly constant. Unlike the ordinary GR case, this result does not apply to the matter-frame potential Φ and the density contrast which are not strictly frozen due to the generally time-varying perturbations of the scalar field. Given the typical magnitudes of scalar field perturbations from numerical analysis, however, the impact on the evolution of Φ and δ will be rather small.

In principle, the analysis presented here can also be performed for more realistic cosmological models, for instance, by including DE effects in terms of a cosmological constant. Having said that, however, such studies are likely to involve a numerical treatment of background quantities. Similar to previous considerations, one might also extend the current parameterization to the background level in these cases. Moreover, note that the basic approach of this work is not limited to the framework of TeVeS. Since the occurrence of trivial identities of the above kind appears as a generic feature of modified gravity theories with extra DOFs, these parameterizations

TABLE II. Summary of relevant perturbation quantities in the conformal Newtonian gauge.

Symbol	Description
Ψ, Φ	Metric potentials in the matter frame
$\tilde{\Psi}, \tilde{\Phi}$	Metric potentials in the Einstein frame
$\tilde{\zeta}$	Einstein-frame metric perturbation, $\tilde{g}_{0i} = -b^2 \partial_i \tilde{\zeta}$
φ	Perturbation of the scalar field ϕ
γ	Auxiliary perturbation related to the gradient of ϕ
α_μ	Perturbation of the vector field A_μ
α	Longitudinal component of the vector field perturbation ^a
E	Auxiliary perturbation related to $F_{\mu\nu}$, E is gauge-invariant
$\delta\rho$	Density perturbation of the matter fluid
δP	Pressure perturbation of the matter fluid
δ	Density contrast of the matter fluid, $\delta = \delta\rho/\bar{\rho}$
θ	Velocity potential of the matter fluid
Σ	Shear of the matter fluid
C_s	Sound speed of the matter fluid, $C_s^2 = \delta P/\delta\rho$
k	Magnitude of the conformal wave vector \mathbf{k} , i.e. $k = \mathbf{k} $
B_φ	New variable absorbing the scalar field DOF, $B_\varphi = \tilde{\Psi}/\varphi$
B_γ	New variable absorbing the vector field DOF, see Eq. (46)

^a Note that α is related to $\tilde{\zeta}$ through Eq. (41).

should generally prove useful when exploring the cosmological properties of other proposed modifications including generalizations of TeVeS and a variety of tensor-vector models.

As such parameterizations capture the full nature of modifications by construction, they further provide an appropriate framework to model quantities like the metric potentials, the gravitational slip or the growth function in a semi-analytic fashion. These models might then be used to constrain modified effects with the help of available cosmological data, but also form a basis for studying other observable imprints like, for instance, the integrated Sachs-Wolfe effect. Supplementary to numerical analysis, parameterizations based on trivial identities thus provide an interesting tool to approach modified gravity theories with extra DOFs.

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Appendix A: Scalar field evolution during tracking

In the following, I will assume the generalized potential defined in Eq. (13) and adopt the notation and definitions used in Ref. [27]. There it has been found that the scalar field evolves during tracking as

$$\bar{\phi} = \bar{\phi}_0 + \phi_1 \log a, \quad (\text{A1})$$

where

$$\phi_1 \equiv \frac{d\bar{\phi}}{d \log a} \quad (\text{A2})$$

is approximately constant. Indeed, following the derivation presented in Ref. [27], one can show that

$$\frac{\phi_1}{1 + \phi_1} = \frac{\beta}{2\mu_0} \sqrt{\left(\frac{1 + 3w}{1 - w}\right)^2}, \quad (\text{A3})$$

where $\beta = \pm 1$ denotes the sign of the scalar field's time derivative, i.e.

$$\beta \equiv \text{sgn} \dot{\bar{\phi}}. \quad (\text{A4})$$

To see that the sign in Eq. (A3) is chosen appropriately, one uses Eq. (A1) and finds that

$$\beta = \text{sgn} \left(\phi_1 \frac{\dot{a}}{a} \right) = \text{sgn} \phi_1 = \text{sgn} \beta, \quad (\text{A5})$$

where I have assumed that $|\phi_1| \ll 1$ for the last equality. Note that this is justified because of the requirement $\mu_0 \gg 1$ for viable cosmological models.

Here the right-hand side of Eq. (A3) deserves special attention: When evaluating the square root, one needs to take into account that the argument's sign does depend on the actual choice of w . Therefore, one has

$$\sqrt{\left(\frac{1 + 3w}{1 - w}\right)^2} = \frac{|1 + 3w|}{|1 - w|}, \quad (\text{A6})$$

which eventually gives the result in Eq. (24). During tracking, the field $\bar{\mu}$ (see Sec. III B) evolves as $\bar{\mu} = 2\mu_0(1 + \epsilon)$, where

$$\log \epsilon \propto -\frac{2\phi_1 + 3(1 + w)}{n} \log a, \quad (\text{A7})$$

and thus $2\phi_1 + 3(1 + w) > 0$ emerges as a condition for stable tracking. For a universe dominated by a cosmological constant Λ , one has $w = -1$ and therefore $\beta = 1$. Since the time derivative of $\bar{\phi}$ changes its sign when passing from the matter to the Λ era (resulting in $\bar{\rho}_\phi$ momentarily going to zero) [29], it follows that $\beta = -1$ during matter domination. This result is in accordance with previous work [11, 34] where it has been shown that $\bar{\phi}$ decreases with time during the matter era.

Appendix B: Field equations in conformal Newtonian gauge

Here I will summarize the resulting TeVeS perturbation equations for scalar modes in the conformal Newtonian gauge. Furthermore, I shall assume a spatially flat spacetime geometry and introduce the fluid's sound speed C_s which is defined as the ratio between the fluid's pressure perturbation δP and the corresponding density perturbation $\delta \rho$, i.e. $C_s^2 = \delta P / \delta \rho$. As usual, the equations are expressed in Fourier space using the conformal wave vector \mathbf{k} in accordance with the coordinate system specified in Eq. (31).

1. Matter fluid equations

The density contrast for scalar modes in the conformal Newtonian gauge evolves as

$$\dot{\delta} = -(1 + w)(k^2 \theta + 3\dot{\Phi}) - 3\frac{\dot{a}}{a}(C_s^2 - w)\delta, \quad (\text{B1})$$

where the velocity potential θ obeys

$$\dot{\theta} = -\frac{\dot{a}}{a}(1 - 3w)\theta + \frac{C_s^2}{1 + w}\delta - \frac{\dot{w}}{1 + w}\theta - k^2 \frac{2}{3}\Sigma + \Psi, \quad (\text{B2})$$

the quantity Σ denotes the shear of the matter fluid and $k = |\mathbf{k}|$. Note that the equations for perturbations of the matter fluid remain unaltered compared to the standard case of GR.

2. Scalar field equation

The perturbed scalar field equation yields

$$\begin{aligned} \dot{\gamma} = & -3\frac{\dot{b}}{b}\gamma + \frac{\bar{\mu}}{a}e^{-3\bar{\phi}}k^2\left(\varphi + \dot{\bar{\phi}}\alpha\right) - 2\frac{\bar{\mu}}{a}e^{\bar{\phi}}\dot{\bar{\phi}}\left(3\dot{\Phi} + k^2\tilde{\zeta}\right) \\ & + 8\pi G a e^{-3\bar{\phi}}\bar{\rho}\left[\left(1 + 3C_s^2\right)\delta + (1 + 3w)(\tilde{\Psi} - 2\varphi)\right] \end{aligned} \quad (\text{B3})$$

and

$$\dot{\varphi} = -\frac{1}{2U}a e^{-\bar{\phi}}\gamma + \dot{\bar{\phi}}\tilde{\Psi}. \quad (\text{B4})$$

Here γ denotes the perturbation of an auxiliary field introduced to split the scalar field equation into two first-order equations [31].

3. Vector field equation

The two first-order equations coming from the perturbed vector equation are

$$\begin{aligned} K_B \left(\dot{E} + \frac{\dot{b}}{b}E \right) = & 8\pi G a^2 \bar{\rho}(1 + w) \left(1 - e^{-4\bar{\phi}} \right) (\theta - \alpha) \\ & - \bar{\mu} \dot{\bar{\phi}} \left(\varphi - \dot{\bar{\phi}}\alpha \right) \end{aligned} \quad (\text{B5})$$

and

$$\dot{\alpha} = E + \tilde{\Psi} + \left(\dot{\bar{\phi}} - \frac{\dot{a}}{a} \right) \alpha, \quad (\text{B6})$$

where the auxiliary scalar mode E is gauge-invariant and related to $F_{\mu\nu}$, the field strength tensor of A_μ [31].

4. Generalized Einstein equations

From the scalar modes of the perturbed generalized Einstein equations, one obtains the Hamiltonian constraint

$$\begin{aligned} -2k^2\tilde{\Phi} - 2e^{4\bar{\phi}}\frac{\dot{b}}{b}\left(3\dot{\Phi} + k^2\tilde{\zeta} + 3\frac{\dot{b}}{b}\tilde{\Psi}\right) \\ + a e^{3\bar{\phi}}\dot{\bar{\phi}}\gamma - K_B k^2 E = 8\pi G a^2 \bar{\rho}(\delta - 2\varphi) \end{aligned} \quad (\text{B7})$$

and the momentum constraint equation

$$\ddot{\Phi} + \frac{\dot{b}}{b}\ddot{\Psi} - \ddot{\mu}\ddot{\phi}\varphi = 4\pi Ga^2 e^{-4\bar{\phi}} \bar{\rho}(1+w)\theta. \quad (\text{B8})$$

Finally, the two propagation equations read

$$\begin{aligned} & 6\ddot{\Phi} + 2k^2(\ddot{\zeta} - e^{-4\bar{\phi}}\ddot{\Psi}) + 2e^{-4\bar{\phi}}k^2\ddot{\Phi} \\ & + 2\frac{\dot{b}}{b}(6\ddot{\Phi} + 3\ddot{\Psi} + 2k^2\ddot{\zeta}) + 4\ddot{\phi}(3\ddot{\Phi} + k^2\ddot{\zeta}) \\ & + 3\frac{\ddot{\mu}}{U}ae^{-\bar{\phi}}\ddot{\phi}\gamma - 6\left(-2\frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} - 4\ddot{\phi}\frac{\dot{b}}{b}\right)\ddot{\Psi} \\ & - 24\pi Ga^2 e^{-4\bar{\phi}}\bar{\rho}(C_s^2\delta - 2w\varphi) = 0 \end{aligned} \quad (\text{B9})$$

and

$$\ddot{\Phi} - \ddot{\Psi} + e^{4\bar{\phi}}\left[\ddot{\zeta} + 2\left(\frac{\dot{b}}{b} + \ddot{\phi}\right)\ddot{\zeta}\right] = 8\pi Ga^2 \bar{\rho}(1+w)\Sigma. \quad (\text{B10})$$

Appendix C: Parameterized perturbation equations

In what follows, I will assume a universe which contains pressureless matter only, in accordance with the modified EdS cosmology introduced in Sec. III C. In this case, the fluid's pressure components may be neglected, i.e. $w = C_s = \Sigma = 0$, and the background density evolves as $\bar{\rho} \propto a^{-3}$.

1. General case

The goal is now to express the metric potentials in terms of matter fluid variables, using the closure relations from Sec. IV B together with the perturbation equations. Since there remain three gravitational fields after applying Eqs. (44) and (46), i.e. the two metric potentials and the longitudinal component α of the vector field perturbation, this requires finding three linearly independent equations. For the first equation, I eliminate the (conformal) time derivative of $\ddot{\zeta}$ between Eqs. (B6) and (B10), which leads to

$$\ddot{\Phi} + (1 - e^{4\bar{\phi}})E - e^{4\bar{\phi}}\ddot{\Psi} - 4\ddot{\phi}\alpha + e^{4\bar{\phi}}\left(\frac{\dot{a}}{a} + 5\ddot{\phi}\right)\ddot{\zeta} = 0. \quad (\text{C1})$$

Differentiating the above and substituting all remaining time derivatives by a suitable combination of the perturbation equations eventually gives

$$\begin{aligned} & -e^{4\bar{\phi}}\frac{B_\varphi}{B_\gamma}k^2\ddot{\zeta} + \left[\frac{(1 - e^{4\bar{\phi}})^2}{K_B}\left(\frac{\ddot{\mu}\ddot{\phi}^2}{1 - e^{4\bar{\phi}}} + 8\pi Ga^2 e^{-4\bar{\phi}}\bar{\rho}\right) - 8\left(1 + 2\frac{1 + e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}\right)\ddot{\phi}^2 - 4\ddot{\phi}\right]\alpha + 4\ddot{\phi}\left(-1 + \frac{1 + e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}\right)\ddot{\Phi} \\ & + e^{4\bar{\phi}}\left[4\frac{1 + e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}\ddot{\phi}\left(\frac{\dot{a}}{a} + 5\ddot{\phi}\right) + \frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2} - 4\ddot{\phi}\frac{\dot{a}}{a} + 5\ddot{\phi}^2 + 5\ddot{\phi}\right]\ddot{\zeta} + 4\pi Ga^2 e^{-4\bar{\phi}}\bar{\rho}\left[1 - 3e^{4\bar{\phi}}\frac{B_\varphi}{B_\gamma} - 2\frac{(1 - e^{4\bar{\phi}})^2}{K_B}\right]\theta \\ & + \left[\left(1 - 3e^{4\bar{\phi}}\frac{B_\varphi}{B_\gamma}\right)\left(\frac{\ddot{\mu}\ddot{\phi}}{B_\varphi} - \frac{\dot{b}}{b}\right) - \frac{1 - e^{4\bar{\phi}}}{K_B}\frac{\ddot{\mu}\ddot{\phi}}{B_\varphi} - e^{4\bar{\phi}}\left(4\frac{1 + e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}\ddot{\phi} + \frac{\dot{B}_\varphi}{B_\varphi} + (4 + B_\varphi)\ddot{\phi}\right) + (1 - e^{4\bar{\phi}})\frac{\dot{b}}{b}\right]\ddot{\Psi} = 0. \end{aligned} \quad (\text{C2})$$

Remembering that $\ddot{\zeta}$ is related to α through Eq. (41), Eq. (C2) obviously connects the perturbations quantities $\ddot{\Psi}$, $\ddot{\Phi}$ and α to the matter perturbation variables (in this case the velocity potential θ only) and the functions B_φ and B_γ . To find a second equation, one may start from Eq. (46). Similar as before, I take its time derivative and use the perturbation equations to recast the resulting expression into a more convenient form. A bit of algebra then reveals

$$\begin{aligned} & -e^{-4\bar{\phi}}k^2\ddot{\Phi} + e^{-4\bar{\phi}}\left(1 + \frac{\ddot{\mu}}{2U}\frac{B_\gamma}{B_\varphi}\right)k^2\ddot{\Psi} + e^{-4\bar{\phi}}\frac{\ddot{\mu}}{2U}B_\gamma\ddot{\phi}k^2\alpha + \left(3\frac{\ddot{\mu}}{B_\gamma}\ddot{\phi} - \frac{\dot{B}_\gamma}{B_\gamma} - \frac{\ddot{\mu}}{U}B_\gamma\ddot{\phi} - 3\frac{B_\varphi}{B_\gamma}\frac{\dot{b}}{b} + \frac{\dot{U}}{U}\right) \\ & \times (k^2\ddot{\zeta} + 12\pi Ga^2 e^{-4\bar{\phi}}\bar{\rho}\theta) + \left[3\left(\frac{\dot{b}}{b} - (4 + B_\varphi)\ddot{\phi} - \frac{\dot{B}_\varphi}{B_\varphi}\right)\frac{\dot{b}}{b} + \frac{4\pi}{U}Ga^2 e^{-4\bar{\phi}}\bar{\rho}\left(B_\gamma - 2\frac{B_\gamma}{B_\varphi}\right) - 6\frac{\dot{b}}{b}\right. \\ & \left. + 3\left(\frac{\ddot{\mu}}{B_\varphi}\ddot{\phi} - \frac{\dot{b}}{b}\right)\left(3\frac{\ddot{\mu}}{B_\gamma}\ddot{\phi} - \frac{\dot{B}_\gamma}{B_\gamma} - \frac{\ddot{\mu}}{U}B_\gamma\ddot{\phi} - 3\frac{B_\varphi}{B_\gamma}\frac{\dot{b}}{b} + \frac{\dot{U}}{U}\right)\right]\ddot{\Psi} + \frac{4\pi}{U}B_\gamma Ga^2 e^{-4\bar{\phi}}\bar{\rho}\delta = 0. \end{aligned} \quad (\text{C3})$$

Finally, the last equation is obtained from eliminating E between Eqs. (C1) and (B7). Together with the relations presented in Sec. IV B, one eventually ends up with

$$\begin{aligned} & -\left(2 - \frac{K_B}{1 - e^{4\bar{\phi}}}\right)k^2\ddot{\Phi} - K_B\frac{e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}k^2\ddot{\Psi} + K_B\left(\frac{\dot{b}}{b} - 4\frac{e^{4\bar{\phi}}}{1 - e^{4\bar{\phi}}}\ddot{\phi}\right)k^2\alpha - 2e^{4\bar{\phi}}\left(\frac{\dot{b}}{b} + \frac{U}{B_\gamma}\ddot{\phi}\right)(k^2\ddot{\zeta} + 12\pi Ga^2 e^{-4\bar{\phi}}\bar{\rho}\theta) \\ & - 8\pi Ga^2 \bar{\rho}\delta - 2e^{4\bar{\phi}}\left[3\frac{\ddot{\mu}U}{B_\varphi B_\gamma}\ddot{\phi}^2 + 3\left(\frac{\ddot{\mu}}{B_\varphi} - \frac{U}{B_\gamma}\right)\frac{\dot{b}}{b}\ddot{\phi} - \frac{8\pi}{B_\varphi}Ga^2 e^{-4\bar{\phi}}\bar{\rho}\right]\ddot{\Psi} = 0. \end{aligned} \quad (\text{C4})$$

As is obvious, the set of Eqs. (C2), (C3) and (C4) forms a closed linear system for the fields $\tilde{\Psi}$, $\tilde{\Phi}$ and α . Thus the corresponding solution will give the fields as expressions of the parameter functions and matter fluid variables only.

2. Superhorizon limit

For scales well outside the horizon, the perturbation equations can be considerably simplified by considering the limit $k \rightarrow 0$, which allows one to begin with Eq. (60). Taking its time derivative, the first step is to eliminate $\dot{\gamma}$ and $\dot{\delta}$ from the resulting equation. This is achieved with the help of

$$\ddot{\Phi} - \left(\frac{\dot{a}}{a} - \frac{\dot{\phi}}{\phi} + \frac{\dot{B}_\gamma}{B_\gamma} - \frac{\dot{U}}{U} \right) \dot{\Phi} + \frac{B_\gamma}{6U} a e^{-\tilde{\phi}} \dot{\gamma} = O(k^2), \quad (C5)$$

which follows from Eq. (46), and the relation Eq. (61), respectively. Then, making use of Eqs. (44) and (46) together with an appropriate combination of the perturbation equations, one can derive an equation governing the evolution of the Einstein-frame potential $\tilde{\Phi}$. Defining the auxiliary quantities

$$\tilde{P} \equiv 3 \frac{\dot{b}^2}{b^2} - 4\pi G a^2 e^{-4\tilde{\phi}} \bar{\rho} \quad (C6)$$

and

$$\tilde{Q} \equiv U \left[2 \frac{\ddot{b}}{b} - \frac{\dot{b}^2}{b^2} + (4 + B_\varphi) \frac{\dot{b}}{b} \frac{\dot{\phi}}{\phi} + \frac{\dot{B}_\varphi}{B_\varphi} \frac{\dot{b}}{b} \right], \quad (C7)$$

the result can be expressed as

$$\begin{aligned} (B_\gamma \tilde{P} + 3\tilde{Q}) \ddot{\Phi} + \left[\tilde{P} \left(2B_\gamma \left(\frac{\dot{b}}{b} + \frac{\dot{\phi}}{\phi} \right) + 3 \left(B_\varphi \frac{\dot{b}}{b} - \frac{\dot{\mu}}{\mu} \right) \right) \right. \\ \left. - 3\tilde{Q} \left(\frac{B_\gamma}{U} \left(\frac{\dot{b}}{b} + \frac{\dot{\mu}}{\mu} \right) - 2 \frac{\dot{b}}{b} - \frac{\dot{\phi}}{\phi} + \frac{\dot{B}_\gamma}{B_\gamma} - \frac{\dot{U}}{U} \right) \right] \dot{\Phi} = 0. \end{aligned} \quad (C8)$$

As is obvious, an immediate solution to the above is given by $\tilde{\Phi} = \text{const}$. Following a similar procedure, it is also possible to derive an equation for the scalar field perturbation φ ,

$$\begin{aligned} B_\varphi \left[\left(\frac{\dot{\mu}}{\mu} + \frac{\dot{b}}{b} \right) \frac{B_\gamma}{U} \dot{\phi} - \left(2 \frac{\dot{b}}{b} + \frac{\dot{\phi}}{\phi} + \frac{\dot{U}}{U} + \frac{\dot{B}_\varphi}{B_\varphi} \right) \dot{\phi} - \frac{\ddot{\phi}}{\phi} - \frac{\ddot{P}}{P} \right] \varphi \\ + \left[2 \frac{\dot{b}}{b} + (1 - B_\varphi) \frac{\dot{\phi}}{\phi} + \frac{\dot{U}}{U} - \left(\frac{\dot{\mu}}{\mu} + \frac{\dot{b}}{b} \right) \frac{B_\gamma}{U} \right] \dot{\varphi} + \ddot{\varphi} = 0, \end{aligned} \quad (C9)$$

which takes the general form of a damped harmonic oscillator and is further discussed in Sec. IV D.

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- [39] Writing TeVeS in pure tensor-vector form, all of these effects can fully be attributed to a single vector field with dynamic norm.